

# MANIFOLD CALCULUS AND HOMOTOPY SHEAVES

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**ABSTRACT.** Manifold calculus is a form of functor calculus concerned with functors from some category of manifolds to spaces. A weakness in the original formulation is that it is not continuous in the sense that it does not handle well the natural enrichments. In this paper, we correct this by defining an enriched version of manifold calculus which essentially extends the discrete setting. Along the way, we recast the Taylor tower as a tower of homotopy sheafifications. As a spin-off we obtain a natural connection to operads: the limit of the Taylor tower is a certain (derived) space of right module maps over the framed little discs operad.

## 1. INTRODUCTION

Let  $M$  be a smooth manifold without boundary and denote by  $\mathcal{O}(M)$  the poset of open subsets of  $M$ , ordered by inclusion. Manifold calculus, as defined in [Wei99], is a way to study (say, the homotopy type of) contravariant functors  $F$  from  $\mathcal{O}(M)$  to spaces which take isotopy equivalences to (weak) homotopy equivalences. In essence, it associates to such a functor a tower - called the *Taylor tower* - of polynomial approximations which in good cases converges to the original functor, very much like the approximation of a function by its Taylor series.

The remarkable fact, which is where the geometry of manifolds comes in, is that the Taylor tower can be explicitly constructed: the  $k^{th}$  Taylor polynomial of a functor  $F$  is a functor  $T_k F$  which is in some sense the universal approximation to  $F$  with respect to the subcategory of  $\mathcal{O}(M)$  consisting of open sets diffeomorphic to  $k$  or less open balls.

A weakness in the traditional discrete approach is that in cases where  $F$  has obvious continuity properties,  $T_k F$  does not obviously inherit them, where by continuous we mean enriched over spaces. For example, let  $F(U)$  for  $U \in \mathcal{O}(M)$  be the space of smooth embeddings from  $U$  to a fixed smooth manifold  $N$ . It is clear that the group of diffeomorphisms  $M \rightarrow M$  acts in a continuous manner on  $F(M)$ . One would expect a similar continuous action of the same group on  $T_k F(M)$ , for all  $k$ . But with the standard description of  $T_k F$  we only get an action of the underlying discrete group which typically is not continuous. As a solution to this problem in the particular case of the embedding functor a continuous model, Haefliger-style, was proposed in [GKW03].

In this paper, we correct this lack of continuity (for every enriched functor) by defining an enriched version of manifold calculus. Along the way, we reapproach the foundations of the theory by focusing on the wider notion of homotopy sheaves rather than on polynomial functors which had a central role in [Wei99]. The sheaf point-of-view was already present in [ibid.] but had a relatively secondary part in that paper, being more of a consequence than a building block *per se*.

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The first author was supported by FCT, Fundação para a Ciência e a Tecnologia, grant SFRH/BD/61499/2009.

We now give a brief overview of the paper. Let  $\mathcal{S}$  be a category of *spaces*, i.e. compactly generated Hausdorff spaces or simplicial sets (more on this at the end of the introduction). To have an enriched setting we replace the category  $\mathcal{O}(M)$  by the category  $\mathcal{E}$  of manifolds of a fixed dimension  $d$  and codimension zero embeddings. We then want to consider (contravariant) functors which are enriched over  $\mathcal{S}$ , namely functors  $F : \mathcal{E}^{op} \rightarrow \mathcal{S}$  inducing *continuous* maps

$$\mathcal{E}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{S}}(F(M), F(N))$$

which preserve the units and composition.

Moreover, there is a natural Grothendieck topology  $\mathcal{J}_1$  on  $\mathcal{E}$  given by usual open covers. For each positive  $k$ , we can define a multi-local version of  $\mathcal{J}_1$  where we only admit covers which have the property that every set of  $k$  (or fewer) points is contained in some open set of the cover. These form the Grothendieck topologies  $\mathcal{J}_k$ .

Since  $\mathcal{E}$  is a site, we refer to  $\mathcal{S}$ -functors on  $\mathcal{E}$  as  $\mathcal{S}$ -enriched presheaves, or simply presheaves.

**Definition 1.1.** The **Taylor tower** of  $F$  is the *tower of homotopy sheafifications* of  $F$  with respect to the Grothendieck topologies  $\mathcal{J}_k$ .

For the precise meaning of this statement, see section 3. The enriched analogue of  $T_k F$  is an  $\mathcal{S}$ -enriched presheaf denoted by  $\mathcal{T}_k F$  (Definition 4.4). It comes with a natural ‘evaluation’ map

$$F \rightarrow \mathcal{T}_k F$$

which we call the *Taylor approximation* of  $F$ . The main result of this paper is

**Theorem 1.2.** *The Taylor approximation is a homotopy  $\mathcal{J}_k$ -sheafification.*

The upshot is that the Taylor tower of  $F$  can be given an explicit model as a tower of homotopical approximations of  $F$  with respect to the subcategories  $\mathcal{E}_k$  of  $\mathcal{E}$  whose objects are disjoint unions of  $k$  or fewer balls. Therefore, the  $k^{th}$  sheafification of  $F$  is essentially determined by the restriction of  $F$  to  $\mathcal{E}_k$  and, as a byproduct, we obtain a very natural connection to operads

$$\mathcal{T}_{\infty} F(M) \simeq \mathbb{R}\mathrm{Hom}_P(\mathrm{Emb}_M, F)$$

where the right hand side is the derived space of right module maps over the framed little discs operad  $P$ , and  $\mathrm{Emb}_M$  and  $F$  are the right  $P$ -modules defined by  $\{\mathrm{Emb}(\amalg_n \mathbb{R}^d, M)\}_{n \geq 0}$  and  $\{F(\amalg_n \mathbb{R}^d)\}_{n \geq 0}$ , respectively. This answers a conjecture of Greg Arone and Victor Turchin (Conjecture 4.14, [AT11]).

In the case where  $F$  is the embedding functor  $\mathrm{Emb}(-, N)$  and  $\dim N - d \geq 3$  we get, as an immediate corollary of Goodwillie-Klein excision estimates, that

$$\mathrm{Emb}(M, N) \simeq \mathbb{R}\mathrm{Hom}_P(\mathrm{Emb}_M, \mathrm{Emb}_N)$$

A discrete version of this connection to operads appeared recently in the work of Arone and Turchin [ibid.] where, coupled with formality results, it is further used to obtain explicit descriptions of the rational homology and homotopy of certain spaces of embeddings.

Finally, we point out that the framework in this paper is rather general and can be applied to other categories other than  $\mathcal{E}$ . Namely, for an  $\mathcal{S}$ -category  $\mathcal{C}$  equipped with a Grothendieck topology possessing *good covers* and, given a presheaf  $F$  on  $\mathcal{C}$ , one can construct the tower of homotopy sheafifications of  $F$  - its Taylor Tower - and give an explicit model for it as a tower of homotopical approximations with respect to certain subcategories of  $\mathcal{C}$ . Examples include the category of  $d$ -dimensional manifolds with boundary, the category of  $d$ -manifolds with boundary with a specified diffeomorphism to the boundary from some fixed manifold  $K$  (c.f. Section 5 and 6 in [RW11]), or the category of all manifolds.

**Outline of the paper.** In section 2 we define homotopy sheaves. We relax the definition of a Grothendieck topology to that of a coverage and we introduce two coverages  $\mathcal{J}_k$  and  $\mathcal{J}_k^\circ$  which in some sense generate the Grothendieck topology  $\mathcal{J}_k$ . We discuss the local model structure on the category of presheaves in section 3. Homotopy sheafification corresponds to a fibrant replacement for this model structure. In section 4, we discuss enriched homotopical Kan extensions. In section 5, we prove the Theorem 1 and, in section 6, we describe  $\mathcal{T}_k F$  as the derived space of right module maps. The equivalence between homotopy sheaves and polynomial functors is treated in section 7. Finally, in section 8 we show that  $\mathcal{T}_k$  is really an ‘enrichment’ of  $T_k$ . Specifically, we show that for functors  $F$  on  $\mathcal{O}(M)$  which, like  $Emb(-, N)$ , factor through  $\mathcal{E}$ , we have a weak equivalence

$$T_k F(U) \simeq \mathcal{T}_k F(U)$$

for every open set  $U$  of  $M$ .

**Spaces, enrichments and notation.** We do not want to be very imposing on which category of spaces we work with. However, we need it to be cartesian closed, considered as enriched over itself and having the usual limits and colimits. The category of compactly generated Hausdorff spaces is a natural candidate and the one we opt for, but everything can easily be formulated simplicially (see Appendix). We denote this category of spaces by  $\mathcal{S}$ . To make  $\mathcal{S}$  enriched over itself give the Hom-sets,  $\text{Hom}_{\mathcal{S}}(X, Y)$ , the (Kelleyfication of the) weak<sup>1</sup> topology (compact-open topology).

Similarly, the category  $\mathcal{E}$  of  $d$ -dimensional manifolds without boundary is enriched over  $\mathcal{S}$ : give the  $C^\infty$  weak topology to the space of smooth embeddings  $\mathcal{E}(M, N)$  and apply Kelleyfication to make it compactly generated and Hausdorff.

All manifolds in this paper are assumed to be paracompact and **without boundary**.

All the categories considered will be enriched over  $\mathcal{S}$ , so we choose not to make this explicit in the notation. Also, we typically write  $\text{Hom}_A(X, Y) \in \mathcal{S}$  when  $A$  is a functor category or the category of spaces, and write  $A(X, Y) \in \mathcal{S}$  otherwise.

**Acknowledgements.** We would like to thank Victor Turchin for numerous helpful comments and corrections on a previous version of this paper. The first author would also like to thank Assaf Libman for a useful discussion on enriched Kan extensions and Mike Shulman for an interesting exchange of emails on higher categories and Kan extensions. Last, but not least, it is a pleasure to acknowledge the influence of the writings of Dan Dugger on this paper (e.g. [DD01] and [DHI04]).

## 2. HOMOTOPY SHEAVES

**Definition 2.1.** Let  $\mathcal{C}$  be a category. A **coverage**  $\tau$  is an assignment to each object  $X \in \mathcal{C}$  of a set  $\text{Cov}_\tau(X)$  of collections of objects in  $\mathcal{C} \downarrow X$ , subject to the following condition: Given  $\mathcal{U} := \{U_i \rightarrow X\}_{i \in I}$  in  $\text{Cov}_\tau(X)$  and a finite subset  $S := \{i_0, \dots, i_n\}$  of  $I$ , the iterated pullback  $U_S := U_{i_0} \times_X \cdots \times_X U_{i_n}$  exists in  $\mathcal{C}$ .

An element  $\mathcal{U} \in \text{Cov}_\tau(X)$  is called a **covering** of  $X$ .

Coverages are cruder than Grothendieck topologies (called pretopologies in [AGV72])). Coverings in a coverage are not required to be stable under pullbacks and composition, for instance. However, they have just enough structure to allow a definition of (homotopy) sheaves.

Let  $(\mathcal{C}, \tau)$  be an  $\mathcal{S}$ -enriched category equipped with a coverage  $\tau$ .

**Definition 2.2.** An  $\mathcal{S}$ -presheaf on  $\mathcal{C}$  (which we will simply call a **presheaf**) is an  $\mathcal{S}$ -enriched functor  $F : \mathcal{C}^{op} \rightarrow \mathcal{S}$ .

<sup>1</sup>The same would not be true if we had taken, for instance, the strong topology.

**Definition 2.3.** A presheaf  $F$  is said to satisfy **descent** for a covering  $\mathcal{U} := \{U_i \rightarrow X\}_{i \in I}$  in  $\tau$  if the natural map

$$F(X) \longrightarrow \operatorname{holim}_{S \subset I} F(U_S)$$

is a weak equivalence. The homotopy limit ranges over all non-empty, finite subsets  $S$  of  $I$  and, for  $S = \{i_0, \dots, i_n\}$ , the object  $U_S$  is the iterated pullback  $U_{i_0} \times_X \cdots \times_X U_{i_n}$ .

A presheaf  $F$  is a **homotopy  $\tau$ -sheaf** (or satisfies  $\tau$ -descent) if it satisfies descent for every covering in  $\tau$ .

*Remark 2.4.* Given a covering  $\mathcal{U}$ , one can form the Čech complex  $\check{\mathcal{U}}$  which is the simplicial space whose  $n$ -simplices are given by  $\prod_{i_0, \dots, i_n} U_{i_0, \dots, i_n}$ . A presheaf  $F$  is then said to satisfy Čech descent if the natural map

$$F(X) \longrightarrow \operatorname{holim}_{\Delta} F(\check{\mathcal{U}})$$

is a weak equivalence (c.f. [DHI04]). This definition is equivalent to our definition 2.2.

**2.1. Coverages on the category of manifolds.** Let  $\mathcal{E}$  be the category of  $d$ -dimensional smooth manifolds and codimension zero embeddings. To ensure we have a small category, we consider its objects to be  $d$ -dimensional smooth submanifolds of  $\mathbb{R}^\infty$ .

Since  $\mathcal{E}$  has pullbacks (which are given by intersection), the condition defining a coverage on this category is vacuous. Manifold calculus provides us with two standard examples of coverages.

**Definition 2.5** (Coverage  $\mathcal{J}_k$ ). The coverings of  $M$  in  $\mathcal{J}_k$  are given by the collection of morphisms in  $\mathcal{E}$  of the form

$$\{f_i : U_i \rightarrow M\}_{i \in I}$$

such that every set of  $k$  or fewer points is contained in  $U_i$  for some  $i \in I$ . These are called  **$k$ -covers**.

Clearly, a 1-cover is the usual notion of an open cover of a manifold.

**Definition 2.6** (Coverage  $\mathcal{J}_k^h$ ). The coverings of  $M$  in  $\mathcal{J}_k^h$  are given by the collection of morphisms in  $\mathcal{E}$  of the form

$$\{f_i : M \setminus A_i \rightarrow M\}_{i \in \{0, \dots, k\}}$$

where  $A_0, \dots, A_k$  are disjoint closed subsets of  $M$ .

Furthermore, we declare the set of coverings of the empty set  $Cov(\emptyset)$  for all our coverages consists of a single element  $\{id : \emptyset \rightarrow \emptyset\}$ . Notice that if we included the covering consisting of the empty set, i.e.  $Cov(\emptyset) = \{\emptyset, id\}$ , then our sheaves would have the property that  $F(\emptyset)$  is contractible.

*Remark 2.7.* A homotopy sheaf for  $\mathcal{J}_k^h$  is usually called a **polynomial functor** of degree  $\leq k$ .

In fact, the coverage  $\mathcal{J}_k$  satisfies the required axioms to be called a Grothendieck topology. The coverage  $\mathcal{J}_k^h$  does not. In any case, it turns out that  $\mathcal{J}_k$  and  $\mathcal{J}_k^h$  have the same homotopy sheaves. This connection was already made in [Wei99] for the unenriched case, but we shall present a new proof in section 7 of this paper.

## 2.2. Generalised good covers.

**Definition 2.8.** Define the full subcategory  $\mathcal{E}_k$  of  $\mathcal{E}$  by

$$Ob(\mathcal{E}_k) := \left\{ \text{manifolds diffeomorphic to } \coprod_{j \leq k} \mathbb{R}^d \text{ for some } 0 \leq j \leq k \right\}$$

and morphism spaces  $\mathcal{E}_k(U, V) := \mathcal{E}(U, V)$ .

*Remark 2.9.* The empty set  $\emptyset$  is an object of  $\mathcal{E}_k$  (this is, by convention, the case  $j = 0$ ).

It was realised long ago that every manifold  $M$  admits a covering  $\{U_i \rightarrow M\}$  such that all finite intersections belong to  $\mathcal{E}_1$  (see, for instance, [BT82], Theorem 5.1). In other words, every manifold can be covered by open balls  $\{U_i\}$  such that every finite non-empty intersection  $U_{i_0} \cap \cdots \cap U_{i_n}$  is again diffeomorphic to an open ball. Bott and Tu called them *good covers*. Good covers clearly define a coverage  $\mathcal{J}_1^\circ$  on  $\mathcal{E}$ .

**Definition 2.10.** A cover  $\{U_i \rightarrow M\}_{i \in I}$  of a manifold  $M$  is called a **good  $k$ -cover** if

- (1) every set of  $k$  or less points is contained in  $U_i$  for some  $i$  in  $I$
- (2) every finite intersection  $U_{i_0} \cap \cdots \cap U_{i_n}$  belongs to  $\mathcal{E}_k$

A good 1-cover is simply a good cover. The multi-local analogue of the paragraph above is

**Proposition 2.11.** *Every manifold  $M$  admits a good  $k$ -cover.*

*Proof.* We assume  $k > 1$ . The case  $k = 1$  is covered by Theorem 5.1 in [BT82]. The subtlety here is that we need to ensure finite intersections of objects in the cover remain elements of  $\mathcal{E}_k$ .

Equip  $M$  with a Riemannian metric  $d$  with positive convexity radius<sup>2</sup>  $R$ . For every  $k$ -tuple of distinct points in  $M$ , i.e.  $x := (x_1, \dots, x_k)$ , choose disjoint geodesically convex neighbourhoods  $U_{x_i}$  of these as follows. Let  $d_x := \inf_{x_i \neq x_j} d(x_i, x_j)$ . Define  $U_{x_i}$  to be the geodesic ball centred at  $x_i$  with diameter  $\text{diam}_x := \min(2R, d_x/100)$  and

$$U_x := U_{x_0} \amalg \cdots \amalg U_{x_k}$$

The collection  $\{U_x : x \text{ is a } k\text{-tuple of distinct points in } M\}$  clearly satisfies condition (1) of definition 2.10. It remains to show that every finite intersection consists of at most  $k$  balls. So let  $x(1), x(2), \dots, x(p)$  be points in  $M^k$  away from the fat diagonal and form the intersection

$$V := U_{x(1)} \cap U_{x(2)} \cap \cdots \cap U_{x(p)}$$

Without loss of generality, suppose  $d_{x(1)}$  is minimal among all  $d_{x(j)}$ , for  $j \leq p$ . We claim that the inclusion induced map

$$\pi_0 V \longrightarrow \pi_0 U_{x(1)}$$

is injective. This is enough because the target set has cardinality exactly  $k$ . Suppose that the map in question is not injective. Then one of the components of  $U_{x(1)}$ , say  $U_{x(1)_1}$ , will contain more than one connected component of  $V$ . It follows easily, by convexity, that  $U_{x(1)_1} \cap U_{x(j)}$  has more than one connected component for some  $j > 1$ . This means that there are at least two distinct balls in  $U_{x(j)}$  whose distance as subsets of  $M$  is at most the diameter of  $U_{x(1)_1}$ . This diameter is less than or equal to  $d_{x(1)}/100$ , and therefore also less than or equal to  $d_{x(j)}/100$ . By construction, however, the distance between any two distinct balls of  $U_{x(j)}$  is at least  $0.99 \times d_{x(j)}$ .  $\square$

**Definition 2.12** (Coverage  $\mathcal{J}_k^\circ$ ). The coverings of  $M$  in  $\mathcal{J}_k^\circ$  are the good  $k$ -covers of  $M$ .

<sup>2</sup>The convexity radius at a point  $x \in M$  is the supremum of all  $r \in \mathbb{R}$  such that  $B_x(r)$ , the geodesic ball centred at  $x$  with radius  $r$ , is geodesically convex. The convexity radius of the manifold  $M$  is the infimum of the convexity radii over all points in  $M$ . Every manifold admits a metric with positive convexity radius, according to [Gre79]. Another important fact is that the convexity radius  $R$  is always less than the injectivity radius, hence a geodesic ball of radius less than  $R$  is diffeomorphic to  $\mathbb{R}^d$  via the exponential map.

### 3. HOMOTOPY SHEAFIFICATION

A coverage can be thought of as a collection of ‘relations’ we want to impose on  $\mathcal{F}$  (for a precise meaning of the word *relations* here, see [DD01]). This leads us to a setting in which we can discuss homotopy sheafifications precisely.

Let  $\mathcal{C}$  be a (small) category enriched over spaces  $\mathcal{S}$  and  $\tau$  a coverage on  $\mathcal{C}$ . Let  $\mathcal{F}$  denote the category of ( $\mathcal{S}$ -enriched) presheaves on  $\mathcal{C}$  with morphisms given by  $\mathcal{S}$ -natural transformations. The category  $\mathcal{F}$  inherits the enrichment of  $\mathcal{S}$ .

**3.0.1. Projective model structure.** The category  $\mathcal{F}$  of presheaves on  $\mathcal{C}$  has a model structure, the so-called projective model structure, where weak equivalences and fibrations in  $\mathcal{F}$  are determined objectwise<sup>3</sup> and cofibrations by a right lifting property with respect to acyclic fibrations. With this structure,

- (1) every presheaf is fibrant (since every object in  $\mathcal{S}$  is fibrant).
- (2) every representable presheaf is cofibrant (by the enriched Yoneda Lemma)

**Definition 3.1.** The derived morphism space is the right derived functor of  $\mathrm{Hom}_{\mathcal{F}}$ ,

$$\mathbb{R}\mathrm{Hom}_{\mathcal{F}}(X, Y) := \mathrm{Hom}_{\mathcal{F}}(QX, Y) \in \mathcal{S}$$

where  $Q$  denotes a cofibrant replacement functor on  $\mathcal{F}$  with the projective model structure.

*Remark 3.2.* The usual caveat applies here: if  $\mathcal{S}$  is chosen to be the category of simplicial sets, then we do need to take an objectwise fibrant replacement of  $Y$ . Since we are working with topological spaces in mind (and every space is fibrant) this is not needed here. See appendix for further details.

**3.0.2. Local model structure.** Homotopy  $\tau$ -sheaves are the ‘local’ objects with respect to the maps of presheaves

$$(3.1) \quad \mathrm{hocolim}_{S \subset I} \mathcal{C}(-, U_S) \rightarrow \mathcal{C}(-, M)$$

for each covering  $\mathcal{U} := \{U_i \rightarrow M\}_{i \in I}$  in  $\tau$ . More precisely,

**Proposition 3.3.** *Homotopy  $\tau$ -sheaves are the presheaves  $F$  for which the map*

$$\mathbb{R}\mathrm{Hom}_{\mathcal{F}}(\mathcal{C}(-, M), F) \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_{\mathcal{F}}(\mathrm{hocolim}_{S \subset I} \mathcal{C}(-, U_S), F)$$

*is a weak equivalence for each covering  $\mathcal{U} := \{U_i \rightarrow M\}_{i \in I}$  in  $\tau$ .*

*Proof.* The homotopy colimit on the right hand side is cofibrant (by [Hir03], Theorem 18.5.2) so we can consider the honest (i.e. non-derived) space of morphisms functor  $\mathrm{Hom}_{\mathcal{F}}$  instead. Moreover,

$$\mathrm{Hom}_{\mathcal{F}}(\mathrm{hocolim}_{S \subset I} \mathcal{C}(-, U_S), F) \simeq \mathrm{holim}_{S \subset I} \mathrm{Hom}_{\mathcal{F}}(\mathcal{C}(-, U_S), F)$$

which one can check by using the usual formulas computing hocolim/holim and cartesian closedness. The assertion now follows by applying the enriched Yoneda Lemma to both sides, namely,

$$\mathrm{Hom}_{\mathcal{F}}(\mathcal{C}(-, X), F) \cong F(X)$$

is a homeomorphism. □

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<sup>3</sup>Meaning that a map of presheaves  $F \rightarrow G$  is said to be a objectwise equivalence (resp. objectwise fibration) if the maps  $F(M) \rightarrow G(M)$  are weak equivalences (resp. fibrations) in  $\mathcal{S}$ , for each  $M \in \mathcal{C}$ .

$$\mathrm{RHom}_{\mathcal{F}}(G, Z) \xrightarrow{\cong} \mathrm{RHom}_{\mathcal{F}}(F, Z)$$

Note that if  $F \rightarrow G$  is an objectwise equivalence, then it is a  $\tau$ -local equivalence.

**Theorem 3.6.** *There is a model structure on the category  $\mathcal{F}$ , called the  $\tau$ -local model structure, in which*

- (1) *the weak equivalences are the  $\tau$ -local equivalences,*
- (2) *the cofibrations are the same as in the projective model structure on  $\mathcal{F}$ ,*
- (3) *the fibrant objects are the homotopy  $\tau$ -sheaves.*

*Proof.* This model structure is the (left) Bousfield localisation of the projective model structure on  $\mathcal{F}$  at the set of all maps of the form (3.1). The statement is then a consequence of the general theory of Bousfield localisations (for more details, see [Hir03]).  $\square$

**Definition 3.7.** A **homotopy  $\tau$ -sheafification** of a presheaf  $F$  is a homotopy  $\tau$ -sheaf  $\tilde{F}$  together with a  $\tau$ -local equivalence  $F \rightarrow \tilde{F}$ .

In other words, sheafification is a fibrant replacement in the  $\tau$ -local model structure on  $\mathcal{F}$ . Two homotopy sheafifications are necessarily weak equivalent by uniqueness (up to weak equivalence) of fibrant replacements.

We will construct an explicit homotopy  $\tau$ -sheafification functor in section 5 when  $\mathcal{C}$  is the category of  $d$ -manifolds and  $\tau$  is  $\mathcal{J}_k$ .

**3.1. Taylor tower.** We return to the category of  $d$ -manifolds  $\mathcal{E}$ . Recall from section 2 that  $\mathcal{E}$  has various topologies  $\mathcal{J}_k$ . By definition, every covering in  $\mathcal{J}_{k+1}$  is a covering in  $\mathcal{J}_k$  so, given a presheaf  $F$  in  $\mathcal{F}$ , we obtain a tower of sheafifications

$$\begin{array}{ccccccc}
 F & & & & & & \\
 \downarrow & \searrow & \searrow & \searrow & \searrow & \searrow & \\
 F^{(0)} & \longleftarrow & F^{(1)} & \longleftarrow & F^{(2)} & \longleftarrow & F^{(3)} & \longleftarrow & \dots
 \end{array}$$

More precisely,

- (1) the map  $F \rightarrow F^{(k)}$  is a homotopy  $\mathcal{J}_k$ -sheafification of  $F$ ,
- (2) the map  $F^{(k+1)} \rightarrow F^{(k)}$  is a homotopy  $\mathcal{J}_k$ -sheafification of  $F^{(k+1)}$ .

This tower is called the *Taylor tower*. The existence of such a tower is guaranteed by the existence of Bousfield localisations in our setting. In section 5 we shall give an explicit model for the Taylor tower. It is clear that any two models are bound to be weakly equivalent by uniqueness of fibrant replacements.

## 4. ENRICHED KAN EXTENSIONS

Recall the category  $\mathcal{E}_k$  whose objects are given by  $k$  or fewer open balls (definition 2.8).

**Definition 4.1.** Let  $\mathcal{F}_k$  denote the category of presheaves on  $\mathcal{E}_k$ , i.e. the objects are  $\mathcal{S}$ -enriched functors  $F : \mathcal{E}_k^{op} \rightarrow \mathcal{S}$  and morphisms are given by  $\mathcal{S}$ -natural transformations.

Let  $i$  be the inclusion  $\mathcal{E}_k \hookrightarrow \mathcal{E}$ . The restriction map  $F \mapsto F \circ i$  fits into an  $\mathcal{S}$ -enriched adjunction

$$\mathcal{F} \rightleftarrows \mathcal{F}_k$$

where the right adjoint, denoted  $Ran_i F$ , is given by the terminal or right Kan extension of  $F \circ i$  along  $i$ . It can be calculated as a weighted end (for details see [Dub70], Thm I.4.3)

$$(Ran_i F)(M) = \int_{U \in \mathcal{E}_k} \text{Hom}_{\mathcal{S}}(\mathcal{E}(U, M), F(U))$$

In other words, it is the equaliser of

$$\prod_{U \in \mathcal{E}_k} \text{Hom}_{\mathcal{S}}(\mathcal{E}(U, M), F(U)) \rightrightarrows \prod_{U, V \in \mathcal{E}_k} \text{Hom}_{\mathcal{S}}(\mathcal{E}_k(U, V) \times \mathcal{E}(V, M), F(U))$$

when evaluated at  $M \in \mathcal{E}$ .

**Proposition 4.2.** *The enriched terminal Kan extension  $Ran_i F$  is  $\mathcal{S}$ -naturally isomorphic to the functor  $M \mapsto \text{Hom}_{\mathcal{F}_k}(\mathcal{E}(-, M), F)$ .*

*Proof.* By direct checking, using the fact that  $\mathcal{S}$  is cartesian closed.  $\square$

*Remark 4.3.* The universal property of the enriched terminal Kan extension is that

$$\text{Hom}_{\mathcal{F}}(G, Ran_i F) \cong \text{Hom}_{\mathcal{F}_k}(G \circ i, F \circ i)$$

natural in  $G \in \mathcal{F}$ . Since  $i$  is a full embedding, one also has that  $Ran_i F(V) \cong F(V)$  for every  $V \in \mathcal{E}_k$  (which can also be obtained from the previous proposition via the enriched Yoneda Lemma). To sum up,  $Ran_i F$  is the best terminal approximation of  $F$  by an  $\mathcal{S}$ -presheaf which agrees with  $F$  on  $\mathcal{E}_k$ .

For homotopy-theoretic purposes this is not appropriate, however. We need to consider the homotopical version of  $Ran_i F$ .

**Definition 4.4.** Let  $F \in \mathcal{F}$ . Define the presheaf  $\mathcal{T}_k F$  in  $\mathcal{F}$  as

$$(\mathcal{T}_k F)(M) := \mathbb{R}\text{Hom}_{\mathcal{F}_k}(\mathcal{E}(-, M), F)$$

*Notation.* Note that we are restricting  $\mathcal{E}(-, M)$  and  $F$  to the subcategory  $\mathcal{E}_k$ ; these are elements of  $\mathcal{F}_k$ . So, strictly speaking,  $\mathcal{T}_k F(M) := \mathbb{R}\text{Hom}_{\mathcal{F}_k}(\mathcal{E}(-, M)|_{\mathcal{E}_k}, F|_{\mathcal{E}_k})$ , although we suppress this redundant information in the notation.

A few comments are in order:

- (1) If  $M$  is in  $\mathcal{E}_k$ , then  $\mathcal{T}_k F(M) \simeq \text{Hom}_{\mathcal{F}_k}(\mathcal{E}_k(-, M), F)$  since representables are cofibrant<sup>4</sup> in the projective model structure on  $\mathcal{F}_k$ . The enriched Yoneda lemma then gives a homeomorphism  $\mathcal{T}_k F(M) \cong F(M)$ . Hence  $\mathcal{T}_k F$  agrees with (meaning, is naturally weakly equivalent to)  $F$  on  $\mathcal{E}_k$ .

<sup>4</sup>Note, however, we are not claiming that there is a *functorial* cofibrant replacement  $Q$  which is the identity on representables.



(2) The adjoint of the evaluation map

$$\begin{aligned} ev : F(M) \times \mathcal{E}(U, M) &\longrightarrow F(U) \\ (x, f) &\mapsto F(f)(x) \end{aligned}$$

gives rise to a morphism  $F \rightarrow \mathrm{Hom}_{\mathcal{F}_k}(\mathcal{E}(-, M), F)$  and, composing with the cofibrant replacement functor  $Q$ , to a morphism

$$\eta : F \longrightarrow \mathcal{T}_k F$$

called the  $k^{\mathrm{th}}$  **Taylor approximation** to  $F$ .

(3) The value of  $\mathcal{T}_k F$  at a manifold  $M$  can be presented as the totalisation of the cosimplicial object given by

$$[j] \mapsto \prod_{U_0, \dots, U_j \in \mathcal{E}_k} \mathrm{Hom}_{\mathcal{S}} \left( \prod_{i=0}^{j-1} \mathcal{E}_k(U_i, U_{i+1}) \times \mathcal{E}(U_j, M), F(U_0) \right)$$

We defer the proof of this fact to the appendix.

*Remark 4.5.* It is natural to ask whether a homotopy (enriched) Kan extension is still a universal solution to an extension problem. The answer is affirmative if we correct the universal property (see [DHKS04] and [Lur09], Proposition A.3.3.12).

In a nutshell, the universal property of a homotopy (enriched) terminal Kan extension of  $Z \in \mathcal{F}_k$  along  $i$  is that there is a natural weak equivalence

$$(4.1) \quad \mathbb{R}\mathrm{Hom}_{\mathcal{F}}(G, \mathbb{R}\mathrm{Ran}_i Z) \simeq \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(G \circ i, Z)$$

for every  $G \in \mathcal{F}_k$ . This is the homotopical and enriched analogue of (4.3).

We note that if we take  $Z$  to be  $F|_{\mathcal{E}_k}$  in the remark above we obtain a universal characterisation of  $\mathcal{T}_k F$  as the presheaf for which there is a canonical weak equivalence

$$(4.2) \quad \mathbb{R}\mathrm{Hom}_{\mathcal{F}}(G, \mathcal{T}_k F) \simeq \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(G|_{\mathcal{E}_k}, F|_{\mathcal{E}_k})$$

natural in  $G \in \mathcal{F}$ .

## 5. A MODEL FOR THE TAYLOR TOWER

This section is the heart of the paper. We show that the Taylor approximation  $\mathcal{T}_k F$  is a model for the homotopy sheafification of  $F$  with respect to  $\mathcal{J}_k$ . This identifies the Taylor tower with the tower of homotopical approximations with respect to the subcategories  $\mathcal{E}_k$  of  $\mathcal{E}$ .

**Theorem 5.1.** *The presheaf  $\mathcal{T}_k F$  is a homotopy  $\mathcal{J}_k$ -sheaf.*

*Proof.* Let  $\{U_i \rightarrow M\}_{i \in I}$  be a  $\mathcal{J}_k$ -cover of  $M$ . For each  $V$  in  $\mathcal{E}_k$ , the spaces  $\mathcal{E}(V, M)$  and  $\mathcal{E}(V, U_S)$  are weakly equivalent to the spaces of framed configurations of  $j$  points in  $M$  and  $U_S$  respectively, where  $j$  is the number of components of  $V$ . Hence, the canonical map of presheaves on  $\mathcal{E}_k$

$$(5.1) \quad \mathrm{hocolim}_{S \subset I} \mathcal{E}(-, U_S) \longrightarrow \mathcal{E}(-, M)$$

is an objectwise equivalence.

Hence,

$$\begin{aligned} \mathcal{T}_k F(M) &\stackrel{\mathrm{def}}{=} \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(\mathcal{E}(-, M), F) \\ &\simeq \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(\mathrm{hocolim}_{S \subset I} \mathcal{E}(-, U_S), F) \\ &\simeq \mathrm{holim}_{S \subset I} \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(\mathcal{E}(-, U_S), F) \\ &= \mathrm{holim}_{S \subset I} \mathcal{T}_k F(U_S) \end{aligned}$$

The first equivalence holds since the derived Hom preserves weak equivalences by definition. The second equivalence follows from Theorem 19.4.4, [Hir03].  $\square$

**Theorem 5.2.** *The following are equivalent for a presheaf  $F \in \mathcal{F}$ .*

- (1)  *$F$  is a homotopy  $\mathcal{J}_k$ -sheaf*
- (2)  *$F$  is a homotopy  $\mathcal{J}_k^\circ$ -sheaf*
- (3) *The  $k^{\text{th}}$  Taylor approximation of  $F$*

$$\eta_M : F(M) \xrightarrow{\simeq} \mathcal{T}_k F(M)$$

*is a weak equivalence for each  $M \in \mathcal{E}$ .*

*Proof.* (1)  $\Rightarrow$  (2) is clear since a good  $k$ -cover is a  $k$ -cover. For (2)  $\Rightarrow$  (3) take a good  $k$ -cover  $\{U_i \rightarrow M\}_{i \in I}$  of  $M$  and let  $F$  be a homotopy  $\mathcal{J}_k^\circ$ -sheaf. We have the following commutative diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{\simeq} & \operatorname{holim}_{S \subset I} F(U_S) \\ \downarrow & & \downarrow \simeq \\ \mathcal{T}_k F(M) & \xrightarrow{\simeq} & \operatorname{holim}_{S \subset I} \mathcal{T}_k F(U_S) \end{array}$$

where the bottom arrow is a weak equivalence by Theorem 5.1 and, by hypothesis, so is the top arrow. The right hand arrow is an equivalence since  $F$  and  $\mathcal{T}_k F$  agree on  $\mathcal{E}_k$  and  $U_S \in \mathcal{E}_k$  by definition of a good  $k$ -cover.

Finally, (3)  $\Rightarrow$  (1) is immediate from Theorem 5.1.  $\square$

**Theorem 5.3.** *The  $k^{\text{th}}$  Taylor approximation of a presheaf  $F$*

$$\eta : F \longrightarrow \mathcal{T}_k F$$

*is a homotopy  $\mathcal{J}_k$ -sheafification.*

*Proof.* In theorem 5.1 we established that  $\mathcal{T}_k F$  is a homotopy  $\mathcal{J}_k$ -sheaf. We now show that the Taylor approximation is a  $\mathcal{J}_k$ -local equivalence.

Let  $Z$  be a homotopy  $\mathcal{J}_k$ -sheaf. By theorem 5.2 the Taylor approximation of  $Z$  is an objectwise equivalence, so we are required to show

$$\mathbb{R}\operatorname{Hom}_{\mathcal{F}}(\mathcal{T}_k F, \mathcal{T}_k Z) \longrightarrow \mathbb{R}\operatorname{Hom}_{\mathcal{F}}(F, \mathcal{T}_k Z)$$

is a weak equivalence. By (4.2), the source and target of this map are weakly equivalent to  $\mathbb{R}\operatorname{Hom}_{\mathcal{F}_k}(F|_{\mathcal{E}_k}, Z|_{\mathcal{E}_k})$ .  $\square$

**Corollary 5.4.** *Let  $\phi : F \rightarrow G$  be a map of homotopy  $\mathcal{J}_k$ -sheaves such that  $\phi|_{\mathcal{E}_k}$  is an objectwise equivalence. Then  $\phi$  is an objectwise equivalence in  $\mathcal{F}$ .*

*Proof.* The statement follows from the commutative diagram below.

$$\begin{array}{ccc} F & \xrightarrow{\phi} & G \\ \simeq \downarrow & & \downarrow \simeq \\ \mathcal{T}_k F & \xrightarrow{\simeq} & \mathcal{T}_k G \end{array}$$

The vertical arrows are weak equivalences by Theorem 5.2. The bottom arrow is a weak equivalence by the universal property of Kan extensions (or by direct checking using the formula defining  $\mathcal{T}_k$ ).  $\square$

**5.1.  $\mathcal{T}_k$ -local structure.** The homotopy idempotent functor  $\mathcal{T}_k : \mathcal{F} \rightarrow \mathcal{F}$  defines yet another model structure on  $\mathcal{F}$  by the Bousfield-Friedlander localisation of the projective model structure (see Section 9 in [Bou01], in particular Theorem 9.3). For this new model structure, which we refer to as the  $\mathcal{T}_k$ -local model structure, a morphism  $Q : F \rightarrow G$  is

- (i) a weak equivalence if the map

$$\mathcal{T}_k Q : \mathcal{T}_k F \rightarrow \mathcal{T}_k G$$

is an objectwise equivalence in  $\mathcal{F}$ .

- (ii) a fibration if it is a objectwise fibration in  $\mathcal{F}$  and the diagram

$$\begin{array}{ccc} F & \xrightarrow{Q} & G \\ \eta \downarrow & & \downarrow \eta \\ \mathcal{T}_k F & \xrightarrow{\mathcal{T}_k Q} & \mathcal{T}_k G \end{array}$$

is a homotopy pullback square.

**Lemma 5.5.** *Suppose  $Q : F \rightarrow G$  is morphism in  $\mathcal{F}$ . Then  $Q$  is  $\mathcal{T}_k$ -local equivalence if and only if it is a  $\mathcal{J}_k$ -local equivalence.*

*Proof.* The morphism  $Q$  is a  $\mathcal{T}_k$ -equivalence if and only if  $F$  agrees (up to weak equivalence) with  $G$  on  $\mathcal{E}_k$ . Equivalently, the map

$$(5.2) \quad \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(G|_{\mathcal{E}_k}, Z|_{\mathcal{E}_k}) \rightarrow \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(F|_{\mathcal{E}_k}, Z|_{\mathcal{E}_k})$$

is a weak equivalence for every presheaf  $Z$ . Since homotopy  $\mathcal{J}_k$ -sheaves are determined by their value on  $\mathcal{E}_k$ , this is also equivalent to the map (5.2) being a weak equivalence for every homotopy  $\mathcal{J}_k$ -sheaf  $Z$ . In other words, given a presheaf  $Z$  there exists a homotopy  $\mathcal{J}_k$ -sheaf which agrees (up to weak equivalence) with  $Z$  on  $\mathcal{E}_k$ , namely  $\mathcal{T}_k Z$ .

Now, let  $Z$  be a homotopy  $\mathcal{J}_k$ -sheaf. By Theorem 5.2 and the universal property of homotopy Kan extensions (4.2), respectively, we obtain the two weak equivalences

$$\mathbb{R}\mathrm{Hom}_{\mathcal{F}}(F, Z) \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_{\mathcal{F}}(F, \mathcal{T}_k Z) \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(F|_{\mathcal{E}_k}, Z|_{\mathcal{E}_k})$$

and similarly for  $G$ .

Therefore  $Q$  being a  $\mathcal{T}_k$ -local equivalence is equivalent to the map

$$\mathbb{R}\mathrm{Hom}_{\mathcal{F}}(G, Z) \longrightarrow \mathbb{R}\mathrm{Hom}_{\mathcal{F}}(F, Z)$$

being a weak equivalence for every homotopy  $\mathcal{J}_k$ -sheaf  $Z$ . □

The two model structures have the same cofibrations by definition and the same weak equivalences by the preceding lemma, so the fibrations coincide.

**Corollary 5.6.** *The  $\mathcal{T}_k$ -local and  $\mathcal{J}_k$ -local model structures on  $\mathcal{F}$  coincide. In particular, the identity functors yield a Quillen equivalence.*

## 6. CONNECTION TO OPERADS

For each positive  $k$ , fix an embedding  $\eta_k$  of the disjoint union of  $k$  copies of  $\mathbb{R}^d$  in  $\mathbb{R}^\infty$ . By taking the images of the embeddings  $\eta_k$  we obtain a category which is topologically isomorphic to  $\mathcal{E}_\infty$ . This category (which we will still refer to as  $\mathcal{E}_\infty$ ) is a topological PROP, i.e. its objects are identified with the non-negative integers, and it has a symmetric monoidal structure (here

given by disjoint union) which corresponds to the addition of integers.  $\mathcal{E}_\infty$  is called *framed little discs* PROP. The framed little discs operad is the part  $\mathcal{E}_\infty(m, 1)$  of the PROP and since

$$\mathcal{E}_\infty(m, n) \cong \coprod_{m_1 + \dots + m_n = m} \mathcal{E}_\infty(m_1, 1) \times \dots \times \mathcal{E}_\infty(m_n, 1)$$

we can reconstruct the PROP from the operad and vice-versa, and so use the two words interchangeably.

Moreover, the category  $\mathcal{F}_\infty$  of  $\mathcal{S}$ -functors on  $\mathcal{E}_\infty^{op}$  is  $\mathcal{S}$ -equivalent to the category of right modules over the framed little discs operad  $P$ , denoted  $\mathbf{Mod}_P$ . Therefore, for a given  $F \in \mathcal{F}$ , we obtain a description of  $\mathcal{T}_\infty F$  as a derived space of right module maps over the framed little discs operad,

$$(6.1) \quad \mathcal{T}_\infty F(M) \simeq \mathbb{R}\mathrm{Hom}_P(\mathrm{Emb}_M, F)$$

where the two obvious right  $P$ -modules are  $\mathrm{Emb}_M(n) := \mathrm{Emb}(\mathrm{II}_n \mathbb{R}^d, M)$  and  $F(n) := F(\mathrm{II}_n \mathbb{R}^d)$ . This answers a conjecture (4.14, [AT11]) of G. Arone and V. Turchin.

Combining (6.1) with the analyticity results of Goodwillie-Klein for the embedding functor one has the following immediate consequence.

**Proposition 6.1.** *Suppose  $\dim(N) - \dim(M) \geq 3$ . Then*

$$\mathrm{Emb}(M, N) \simeq \mathbb{R}\mathrm{Hom}_P(\mathrm{Emb}_M, \mathrm{Emb}_N)$$

*Remark 6.2.* For finite  $k$ , we obtain ‘truncated’ versions of (6.1). The category  $\mathcal{F}_k$  of  $\mathcal{S}$ -enriched presheaves on  $\mathcal{E}_k$  is  $\mathcal{S}$ -equivalent to the category of  $k$ -truncated right modules over the  $k$ -truncated framed little discs operad. The composition product on the the category of  $k$ -truncated (symmetric) sequences is the obvious one,

$$M(n) \times M(m_1) \times \dots \times M(m_n) \rightarrow M(m_1 + \dots + m_n)$$

only defined when  $m_1 + \dots + m_n \leq k$ .

Let  $\mathrm{Emb}_M(n) := \mathrm{Emb}(\mathrm{II}_n \mathbb{R}^d, M)$  and  $F(n) := F(\mathrm{II}_n \mathbb{R}^d)$  for  $n \leq k$  be  $k$ -truncated sequences of spaces. Then  $\mathrm{Emb}_M$  and  $F$  are  $k$ -truncated modules over the  $k$ -truncated framed little discs operad  $P_k := \{P(n)\}_{n \leq k}$ , and we see that

$$(6.2) \quad \mathcal{T}_k F(M) \simeq \mathbb{R}\mathrm{Hom}_{P_k}(\mathrm{Emb}_M, F)$$

Another example of interest is the singular chains of the embedding functor,  $S_* \mathrm{Emb}(-, N)$ . We will briefly sketch how to obtain a chain complex version of  $\mathcal{T}_k$ . We write  $S_*$  for the normalised singular chains functor  $Top \rightarrow Ch_{\geq 0}$ . Since it is a lax monoidal functor, we can use it to enrich  $\mathcal{E}$  over chain complexes. We denote by  $\mathcal{E}(M, N)_*$  the chain complex  $S_* \mathcal{E}(M, N)$  of morphisms between  $M$  and  $N$  in  $\mathcal{E}$ .

Rename, for the rest of this section,  $\mathcal{F}$  (resp.  $\mathcal{F}_k$ ) as the category of  $Ch_{\geq 0}$ -enriched presheaves from  $\mathcal{E}$  (resp.  $\mathcal{E}_k$ ) to  $Ch_{\geq 0}$  and define

$$\mathcal{T}_k^{Ch} F(M) := \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(\mathcal{E}(-, M)_*, F) \in Ch_{\geq 0}$$

The arguments of the previous sections show that  $F \rightarrow \mathcal{T}_k^{Ch} F$  is a homotopy  $\mathcal{J}_k$ -sheafification. It is also not hard to show that  $\mathcal{F}_k$  is  $Ch_{\geq 0}$ -equivalent to the category of right modules over  $S_* P$ , the chains of the framed little discs operad. Hence,

$$\mathcal{T}_\infty^{Ch} F(M) \simeq \mathbb{R}\mathrm{Hom}_{S_* P}(S_* \mathrm{Emb}_M, F)$$

In the particular case when  $F$  is the (normalised) singular chains of  $\mathrm{Emb}(-, N)$  we obtain the following result, by the analyticity results in [Wei04].

**Proposition 6.3.** *Suppose  $2 \dim(M) + 1 \leq \dim(N)$ . Then there is a chain homotopy equivalence*

$$S_* \text{Emb}(M, N) \simeq \mathbb{R}\text{Hom}_{S_* P}(S_* \text{Emb}_M, S_* \text{Emb}_N)$$

## 7. HOMOTOPY $\mathcal{J}_k$ -SHEAF = POLYNOMIAL FUNCTOR

Recall from section 2 that a polynomial functor of degree  $\leq k$  is a homotopy sheaf for the coverage  $\mathcal{J}_k^h$ .

**Definition 7.1.** A presheaf  $F \in \mathcal{F}$  is **good** if, for any sequence  $U_0 \subset U_1 \subset \dots$  in  $\mathcal{E}$  whose union is  $M$ , the natural map

$$F(M) \longrightarrow \text{holim}_i F(U_i)$$

is a weak equivalence of spaces.

**Theorem 7.2.** *The following are equivalent*

- (1)  *$F$  is a homotopy  $\mathcal{J}_k$ -sheaf*
- (2)  *$F$  is good and polynomial of degree  $\leq k$ .*

*Proof.* A covering in  $\mathcal{J}_k^h$  is a covering in  $\mathcal{J}_k$  so in order to show (1)  $\Rightarrow$  (2) we need only to prove goodness. Observe that a covering  $\{U_i \rightarrow M\}_{i \in \mathbb{N}}$  of  $M$  with  $U_i \subset U_{i+1}$  is a  $k$ -cover and

$$\text{holim}_{S \subset I} F(U_S) \simeq \text{holim}_i F(U_i)$$

so the homotopy  $\mathcal{J}_k$ -sheaf property for these coverings is precisely the condition of goodness.

Now, suppose  $F \in \mathcal{F}$  is good and polynomial of degree  $\leq k$ . By Theorem 5.2, we are required to show that

$$(7.1) \quad F(M) \rightarrow \mathcal{T}_k F(M)$$

is a weak equivalence for every  $M \in \mathcal{E}$ . The proof is now essentially the same as the one of Theorem 5.1 in [Wei99].

Due to the goodness of  $F$ , we can assume that  $M$  is the interior of a compact handlebody  $L$ . Take a handle decomposition of  $L$  with top-dimensional handles of index  $s$ .

The first case is  $s = 0$ , i.e.  $M \simeq \Pi_i \mathbb{R}^d$  for some  $i$ . If  $i \leq k$ , then  $F(M) \simeq \mathcal{T}_k F(M)$  since  $F$  and  $\mathcal{T}_k F$  agree on  $\mathcal{E}_k$  by construction. For  $i > k$  we proceed inductively. Choose  $k + 1$  distinct components  $A_0, \dots, A_k$  of  $M$  and consider the commutative diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{\eta} & \mathcal{T}_k F(M) \\ \simeq \downarrow & & \downarrow \simeq \\ \text{holim}_{S \subset \{0, \dots, k\}} F(M \setminus A_S) & \xrightarrow{\simeq} & \text{holim}_{S \subset \{0, \dots, k\}} \mathcal{T}_k F(M \setminus A_S) \end{array}$$

The vertical arrows are weak equivalences since  $F$  is polynomial of degree  $\leq k$  by hypothesis, and the horizontal arrow is a weak equivalence by induction.

Now, suppose  $s > 0$ . Pick one of the  $s$ -handles

$$e : D^{d-s} \times D^s \longrightarrow L$$

where  $e^{-1}(\partial L) = D^{d-s} \times S^s$ .

Take pairwise disjoint closed discs  $C_0, \dots, C_k$  in  $D^s$  and define

$$A_i := e(D^{d-s} \times C_i) \cap M$$

Then,

- (1)  $A_i$  is closed in  $M$  and  $M \setminus A_i$  is the interior of a smooth handlebody with a handle decomposition with fewer s-handles, and so is any intersection of these,  $\cap_{i \in S} M \setminus A_i$ , for  $S$  a non-empty subset of  $\{0, \dots, k\}$ .
- (2) The family  $\{M \setminus A_i \rightarrow M\}_{i \in \{0, \dots, k\}}$  is a covering of  $M$  for the coverage  $\mathcal{J}_k^h$ .

By induction, as in the case  $s = 0$ , the statement is easily verified.  $\square$

*Remark 7.3.* A way to paraphrase the proposition above is as follows. Define a coverage by declaring  $\text{Cov}(X)$  to consist of coverings in  $\mathcal{J}_k^h$  and coverings of the form  $\{U_i \rightarrow M\}_{i \in \mathbb{N}}$  with  $U_i \subset U_{i+1}$ . Then Proposition 7.2 says that this coverage and  $\mathcal{J}_k$  have the same sheaves.

Polynomial functors are important in manifold calculus (and in functor calculus in general) as they can be given rather explicit descriptions in terms of cubical diagrams. The coverings in  $\mathcal{J}_k^h$  are in practice much smaller than arbitrary or good  $k$ -covers so they are easier to handle.

**Example 7.4.** A polynomial functor of degree  $\leq 1$  is a functor  $F$  which sends homotopy pushout squares to homotopy pullback squares. These are also called linear functors.

**7.1. Classification of linear functors.** Following Goodwillie, we call a presheaf  $F$  *reduced* if  $F(\emptyset) \simeq *$ . If  $F$  is reduced, then

$$\begin{aligned} \mathcal{T}_1 F(M) &\simeq \text{Hom}^{hO(d)}(\mathcal{E}(\mathbb{R}^d, M), F(\mathbb{R}^d)) \\ &\simeq \text{Hom}^{O(d)}(\text{frame}(M), F(\mathbb{R}^d)) \\ &\simeq \Gamma(\text{frame}(M) \times_{O(d)} F(\mathbb{R}^d) \rightarrow M) \end{aligned}$$

where  $\text{frame}(M)$  denotes the total space of the tangent frame bundle of  $M$ ,  $\text{Hom}^{O(d)}(-, -)$  the space of  $O(d)$ -maps and  $\text{Hom}^{hO(d)}(-, -)$  its derived functor. The first equivalence above follows from the fact that  $\mathcal{E}_1(\mathbb{R}^d, \mathbb{R}^d) := \text{Emb}(\mathbb{R}^d, \mathbb{R}^d) \simeq O(d)$  and  $F(\emptyset) \simeq *$ .

Therefore, a reduced presheaf  $F$  is a homotopy  $\mathcal{J}_1$ -sheaf (i.e. linear and good) if and only if

$$F(M) \xrightarrow{\simeq} \Gamma(\text{frame}(M) \times_{O(d)} F(\mathbb{R}^d) \rightarrow M)$$

is a weak equivalence for every  $M$  in  $\mathcal{E}$ . This map is sometimes referred to as *Segal's scanning map*.

*Remark 7.5.* Not every homotopy  $\mathcal{J}_1$ -sheaf is reduced: take for instance a constant sheaf.

## 8. RELATION TO THE UNENRICHED MODEL

Let  $\mathcal{O}(M)$  be the poset of open sets of a manifold  $M$ , i.e. the discrete version of  $\mathcal{E}$ . Clearly there is an ‘inclusion’ functor

$$\mathcal{O}(M) \rightarrow \mathcal{E}$$

given by inclusion  $\text{Ob}(\mathcal{O}(M)) \hookrightarrow \text{Ob}(\mathcal{E})$  on object-sets and sending a morphism  $U \subset V$  in  $\mathcal{O}(M)$  to an embedding  $i \in \mathcal{E}(U, V)$  representing it.

**Definition 8.1.** A functor  $f : \mathcal{O}(M)^{\text{op}} \rightarrow \mathcal{S}$  is called *context-free*<sup>5</sup> if it factors through  $\mathcal{E}$  by an  $\mathcal{S}$ -functor  $F : \mathcal{E}^{\text{op}} \rightarrow \mathcal{S}$ .

The discrete analogues of  $\mathcal{E}_k$  and  $\mathcal{T}_k$  are denoted  $\mathcal{O}_k$  and  $T_k$  respectively. We refer the reader to [Wei99] for details on the unenriched setting. The following proposition says that  $\mathcal{T}_k$  is really an enrichment of  $T_k$ .

<sup>5</sup>This terminology is due to G. Arone and V. Turchin

**Proposition 8.2.** *Let  $f$  be a context-free functor on  $\mathcal{O}(M)$ . Then*

$$T_k f(U) \simeq \mathcal{T}_k F(U)$$

for every  $U \in \mathcal{O}(M)$ .

*Proof.* By definition,  $T_k f(U) := \operatorname{holim}_{V \in \mathcal{O}_k(U)} f(V)$ . Then,

$$\begin{aligned} T_k f(M) &= \operatorname{holim}_{V \in \mathcal{O}_k(M)} F(V) \\ &\cong \operatorname{holim}_{V \in \mathcal{O}_k(M)} \operatorname{Hom}_{\mathcal{F}_k}(\mathcal{E}_k(-, V), F) \\ &\simeq \operatorname{Hom}_{\mathcal{F}_k}(\operatorname{hocolim}_{V \in \mathcal{O}_k(M)} \mathcal{E}_k(-, V), F) \\ &\simeq \mathbb{R}\operatorname{Hom}_{\mathcal{F}_k}(\mathcal{E}(-, M), F) \end{aligned}$$

The first weak equivalence holds because  $f$  is context-free. The second is the enriched Yoneda lemma. The last equivalence follows from the fact that the map of presheaves in  $\mathcal{F}_k$

$$(8.1) \quad \operatorname{hocolim}_{V \in \mathcal{O}_k(M)} \mathcal{E}_k(-, V) \rightarrow \mathcal{E}(-, M)$$

is an objectwise equivalence since, again,  $\mathcal{E}_k(-, V)$  and  $\mathcal{E}_k(-, M)$  are framed configuration spaces. Moreover, the left-hand side is cofibrant in the projective model structure since representables  $\mathcal{E}_k(-, V)$  are cofibrant and the homotopy colimit of an objectwise cofibrant diagram in a simplicial model category is cofibrant by Theorem 18.5.2, [Hir03].  $\square$

## APPENDIX A. DERIVED MAPPING SPACES AND RESOLUTIONS

The categories  $\mathcal{F}$  and  $\mathcal{F}_k$  are endowed with the projective model structure throughout this appendix. Recall this means that weak equivalences and fibrations are determined objectwise.

**A.1. Derived mapping spaces.** We now want to make a distinction between simplicial sets and compactly generated Hausdorff spaces (CGHS). The category  $\mathcal{F}$  of  $\mathcal{S}$ -enriched functors on  $\mathcal{E}$  is  $\mathcal{S}$ -enriched. Given  $X$  and  $Y$ , we denote its enriching morphism object by

$$\operatorname{Hom}_{\mathcal{F}}(X, Y)$$

If  $\mathcal{S}$  = simplicial sets, then  $\operatorname{Hom}_{\mathcal{F}}(X, Y)$  is the simplicial set whose set of  $n$ -simplices is given by set of natural transformations  $X \otimes \Delta[n] \rightarrow Y$ , where  $(X \otimes \Delta[n])(U) := X(U) \times \Delta[n]$ . This makes  $\mathcal{F}$  into a simplicial model category.

If  $\mathcal{S}$  = CGHS, then  $\operatorname{Hom}_{\mathcal{F}}(X, Y)$  denotes the space we obtain by topologising the set of natural transformations  $X \rightarrow Y$  by the subspace topology of the product

$$\prod_{U \in \mathcal{E}} \operatorname{Hom}_{\mathcal{S}}(X(U), Y(U))$$

equipped with the product topology.

*Remark A.1.* Replacing  $\mathcal{E}$  by  $\mathcal{E}_k$ , we enrich  $\mathcal{F}_k$  over  $\mathcal{S}$  in the same manner.

**Definition A.2.** The derived mapping space functor is the right derived functor of  $\operatorname{Hom}_{\mathcal{F}}$ ,

$$\mathbb{R}\operatorname{Hom}_{\mathcal{F}}(X, Y) := \operatorname{Hom}_{\mathcal{F}}(QX, RY) \in \mathcal{S}$$

where  $Q$  and  $R$  denote, respectively, cofibrant and fibrant replacement functors.

*Remark A.3.* If  $\mathcal{S}$  is the category of CGHS then, since every object in  $\mathcal{F}$  is fibrant,  $\mathbb{R}\operatorname{Hom}_{\mathcal{F}}(X, Y) = \operatorname{Hom}_{\mathcal{F}}(QX, Y)$ . The analogous definition of derived mapping space functor applies in the subcategory  $\mathcal{F}_k$ , namely  $\mathbb{R}\operatorname{Hom}_{\mathcal{F}_k}(X, Y) = \operatorname{Hom}_{\mathcal{F}_k}(Q'X, Y)$ , where  $Q'$  denotes a cofibrant replacement functor on  $\mathcal{F}_k$ .

If  $\mathcal{S}$  is CGHS, we can still define a simplicial model structure on  $\mathcal{F}$ . The simplicial set of natural transformations between  $X$  and  $Y$ , which we denote by  $\text{Map}(X, Y)$ , has the following set of  $n$ -simplicies

$$\text{Map}(X, Y)_n := \text{Hom}_{\mathcal{F}}(X \otimes \Delta[n], Y)$$

where  $(X \otimes \Delta[n])(U) := X(U) \times |\Delta[n]|$ ,  $U \in \mathcal{E}$ .

These two enrichments are related by

$$\text{Map}(X, Y) \simeq \text{Sing}(\text{Hom}_{\mathcal{F}}(X, Y))$$

The ‘derived’ version of  $\text{Map}$  is the homotopy function complex of Dwyer-Kan, denoted by  $\text{hMap}$ . In fact, since  $\text{Map}$  makes  $\mathcal{F}$  into a simplicial model category, a model for  $\text{hMap}$  is given ([DK80], Cor. 4.7) by  $\text{Map}(QX, Y)$  where  $Q$  denotes a cofibrant replacement functor on  $\mathcal{F}$ . Moreover,  $\text{hMap}(X, Y) \simeq \text{Sing}(\mathbb{R}\text{Hom}_{\mathcal{F}}(X, Y))$ .

**A.2. Resolutions.** In this section we discuss the construction of a resolution of a presheaf  $F \in \mathcal{F}_k$ . More precisely, we wish to find a cofibrant presheaf  $\tilde{F}$  and a weak equivalence  $\tilde{F} \rightarrow F$ , where everything in sight should be enriched as always.

**A.2.1. Free presheaves on  $\mathcal{E}_k$ .** Let  $\mathcal{E}_k^\delta$  denote<sup>6</sup> the category with the same objects as  $\mathcal{E}_k$  but only identity morphisms. Define  $\mathcal{F}_k^\delta$  to be the category of contravariant functors from  $\mathcal{E}_k^\delta$  to  $\mathcal{S}$  and consider the following free-forgetful adjunction

$$(A.1) \quad L : \mathcal{F}_k^\delta \rightleftarrows \mathcal{F}_k : U$$

where  $U$  is the obvious forgetful functor and  $L$  is a left adjoint to  $U$ . In other words,  $L(Z)$  is the (enriched) left Kan extension of a discrete presheaf  $Z \in \mathcal{F}_k^\delta$  along the inclusion  $i : \mathcal{E}_k^\delta \hookrightarrow \mathcal{E}_k$ . More concretely,

$$L(Z) = \mathcal{E}_k(-, i) \otimes_{\mathcal{E}_k^\delta} Z = \int^{U \in \mathcal{E}_k^\delta} \mathcal{E}_k(-, i(U)) \times Z(U) = \coprod_{U \in \mathcal{E}_k} \mathcal{E}_k(-, U) \times Z(U)$$

Also note that the free-forgetful adjunction  $(U, L)$  is enriched so, in particular, we obtain homeomorphisms

$$\text{Hom}_{\mathcal{F}_k}(L(Z), F) \cong \text{Hom}_{\mathcal{F}_k^\delta}(Z, F \circ i) = \coprod_{U \in \mathcal{E}_k} \text{Hom}_{\mathcal{S}}(Z(U), F(U))$$

which are  $\mathcal{S}$ -natural in  $F$  and  $Z$ .

**A.2.2. Cotriple resolution.** Associated to the free-forgetful adjunction we construct a simplicial object in  $\mathcal{F}_k$ , usually called the cotriple resolution.

For  $Z$  in  $\mathcal{F}_k$ , we define  $\mathcal{L}(Z)_\bullet$  as follows

$$\mathcal{L}(Z)_n := (LU)^{n+1}(Z) \in \mathcal{F}_k$$

Note that  $\mathcal{L}(Z)_\bullet$  is naturally augmented via the map  $LU(Z) \rightarrow Z$  given by the composition (remember  $Z$  is contravariant)

$$\begin{array}{ccc} \mathcal{E}_k(V, U) \times Z(U) & \longrightarrow & Z(V) \\ (f, x) & \longmapsto & Z(f)(x) \end{array}$$

Finally, we define  $|Z| \in \mathcal{F}_k$  to be the geometric realisation of  $\mathcal{L}(Z)_\bullet$ ,

$$|Z| := |\mathcal{L}(Z)_\bullet|$$

**Theorem A.4.** *The presheaf  $|Z|$  is a cofibrant replacement of  $Z$  in  $\mathcal{F}_k$ .*

<sup>6</sup>The symbol  $\delta$  stands for ‘discrete’.



*Proof.* Firstly, the natural map  $|Z| \rightarrow Z$  is a weak equivalence by general considerations of cotriple resolutions. Moreover,  $|Z|$  is cofibrant since  $\mathcal{L}(Z)_\bullet$  is Reedy cofibrant and geometric realisation preserves cofibrations.  $\square$

**Proposition A.5.**  $\mathrm{Hom}_{\mathcal{F}_k}(|Z|, F) \simeq \mathrm{Tot} \mathrm{Hom}_{\mathcal{F}_k}(\mathcal{L}(Z)_\bullet, F)$ , for  $F$  and  $Z$  in  $\mathcal{F}_k$ .

Hence, by Theorem A.4 and Proposition A.5,

$$\begin{aligned} \mathcal{T}_k F(M) &\stackrel{\text{def}}{=} \mathbb{R}\mathrm{Hom}_{\mathcal{F}_k}(\mathcal{E}(-, M), F) \\ &\simeq \mathrm{Hom}_{\mathcal{F}_k}(|\mathcal{E}(-, M)|, F) \\ &\simeq \mathrm{Tot} \mathrm{Hom}_{\mathcal{F}_k}(\mathcal{L}(M)_\bullet, F) \end{aligned}$$

where, for ease of notation, we write  $\mathcal{L}(M)_\bullet$  to mean  $\mathcal{L}(\mathcal{E}(-, M))_\bullet$ , the cotriple resolution associated to  $\mathcal{E}(-, M)$ .

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